

Properties of Moyal-Lax Representation

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Abstract

The properties of standard and the nonstandard Moyal-Lax representations are systematically investigated. It is shown that the Moyal-Lax equation can be interpreted as a Hamiltonian equation and can be derived from an action. We show that the parameter of non-commutativity, in this case, is related to the central charge of the second Hamiltonian structure of the system. The Moyal-Lax description leads in a natural manner to the dispersionless limit and provides the second Hamiltonian structure of dispersionless integrable models, which has been an open question for sometime.

I. Introduction:

Integrable models [1], both bosonic as well as supersymmetric [2], have played important roles in the study of conformal field theories, strings, membranes as well as topological field theories. In recent years, it has become known that string (membrane) theories naturally lead to non-commutative field theories [3], where usual multiplication of functions is replaced by the star product due to Groenewold [4], [5]. It is interesting, therefore, to ask if integrable systems can also be described in terms of star products and Moyal brackets [5]. It appears that there are two possible approaches to this problem. In the first method [6] one can formulate the soliton theory in the non-commutative space-time which is realized using the star product. In the second approach, the star product can be used directly in the Lax operator description [7]-[8] or in the zero-curvature condition [9]-[10].

In this letter, we follow the second approach and show that it is possible to use the Moyal bracket as the Poisson bracket in the phase space. Such an interpretation allows us to present a Moyal-Lax representation for the soliton system and study such representations systematically. So far, the Moyal bracket has been used to construct only the standard Lax representation for bosonic integrable systems [7]. We show that such brackets can be used to describe nonstandard representations as well. There are many interesting features that emerge from such a representation and we argue that such a representation may, in fact, be more desirable. Among the various interesting features that emerge, we note that the parameter of non-commutativity, in such an analysis, is directly related to the central charge of the second Hamiltonian structure of the system. Furthermore, in such a formulation, the Lax equation has a natural interpretation of a Hamiltonian equation and can be simply derived from an action. Furthermore, such a description naturally leads to the dispersionless limit (in which the models become related to membranes and topological field theories) and thereby allows us to derive the Hamiltonian structures (first, second,...), which has been an open question for quite some time.

II. Basic Definitions:

Integrable systems are Hamiltonian systems and, therefore, are naturally defined on a phase space. The star product of two functions, on this space, is defined to be

$$A(x, p) \star B(x, p) = e^{\kappa(\partial_x \partial_{\tilde{p}} - \partial_p \partial_{\tilde{x}})} A(x, p) B(\tilde{x}, \tilde{p}) \Big|_{\tilde{x}=x, \tilde{p}=p}. \quad (1)$$

The conventional Moyal bracket, then, follows to be

$$\left\{ A(x, p), B(x, p) \right\}_{\kappa} = \frac{1}{2\kappa} (A \star B - B \star A). \quad (2)$$

Here κ is the parameter of non-commutativity, which, as we will see, is directly related to the central charge of the second Hamiltonian structure in the case of integrable models. From (1) and

(2), it follows immediately that

$$\lim_{\kappa \rightarrow 0} \{A, B\}_\kappa = \{A, B\}, \quad (3)$$

where the last bracket in (3) stands for the standard canonical Poisson bracket.

The star product gives the momentum an operator character. In particular, let us note that for any arbitrary integer m, n (positive or negative)

$$\begin{aligned} p^n \star p^m &= p^{n+m} \\ p^n \star f(x) &= \sum_{m=0}^n \binom{n}{m} (-2\kappa)^m f^{(m)} \star p^{n-m}, \end{aligned} \quad (4)$$

where

$$\binom{n}{m} = \frac{n(n-1)\dots(n-m+1)}{m!}, \quad \binom{n}{0} = 1, \quad (5)$$

and $f^{(m)}(x)$ stands for the m -th derivative of $f(x)$ with respect to x . We note that these are precisely the relations (up to normalizations) satisfied by the derivative operator.

With these, we can define two classes of Lax operators on the phase space as

$$\begin{aligned} L_n &= p^n + u_1(x) \star p^{n-1} + u_2(x) \star p^{n-2} + \dots + u_n(x) \\ \Lambda_n &= p^n + u_1(x) \star p^{n-1} + \dots + u_n(x) + u_{(n+1)} \star p^{-1} + \dots, \end{aligned} \quad (6)$$

which we can identify respectively with the Lax operator for the generalized KdV hierarchy and the KP hierarchy [8]. In simple terms, we have replaced the space of pseudo-differential operators by polynomials in momentum, which nonetheless inherit an operator structure through the star product and define an algebra. We will call such an algebra the Moyal momentum algebra (Mm algebra)¹. It is easy to check that all the known properties of pseudo-differential operators carry through with suitable redefinitions. For example, for any two arbitrary operators A and B , which are elements of the Mm algebra, the residue (the coefficient of the p^{-1} term with respect to the Moyal product) of the Moyal bracket can be checked to be a total derivative, namely

$$Res\{A, B\}_\kappa = (\partial_x C). \quad (7)$$

Consequently, one can define

$$Tr A = \int dx \, Res \, A, \quad (8)$$

which is unique (with the usual assumptions of asymptotic fall off) and which satisfies cyclicity.

For a general Lax operator Λ_n , it is straightforward to show that

$$\frac{\partial \Lambda_n}{\partial t_k} = \left\{ \Lambda_n, \left(\Lambda_n^{\frac{k}{n}} \right)_{\geq m} \right\}_\kappa, \quad k \neq ln, \quad (9)$$

¹Notice that our Mm algebra is different from the concept of pseudo-differential operators with the coefficients taken from the Moyal algebra introduced in [12].

where k, l are integers, defines a consistent Lax equation, provided $m = 0, 1, 2$ and that the projections are defined with respect to the star product. Note here that $A^{\frac{k}{n}} = A^{\frac{1}{n}} \star A^{\frac{1}{n}} \star \dots \star A^{\frac{1}{n}}$ involving k such factors and the n -th root is determined formally in a recursive manner. The projection with $m = 0$ will be denoted by $(\cdot)_+$ and the corresponding equation will be known as the standard Moyal-Lax representation, while the other two cases will be known as non-standard representations. Let us note the important property

$$\lim_{\kappa \rightarrow 0} \left(\Lambda \star \Lambda' \right)_{\geq m} = \left(\Lambda \Lambda' \right)_{\geq m}, \quad (10)$$

where the factors on the right hand side are functions on the phase space (not operators). One of the advantages of this method is now obvious, namely, one can go to the Lax representation of the model in the dispersionless limit in a natural manner [13]. In fact, if we consider the limit $\kappa \rightarrow 0$, in such a model, it leads to

$$\frac{\partial \Lambda_n}{\partial t} = \left\{ \Lambda_n, \left(\Lambda_n^{\frac{k}{n}} \right)_{\geq m} \right\}. \quad (11)$$

where the bracket on the right hand side is the standard Poisson bracket. With these definitions, one can construct the conserved charges as

$$H_k = Tr \Lambda_n^{\frac{k}{n}}, \quad k \neq ln, \quad (12)$$

where k, l are integers, prove that different flows commute and can define Hamiltonian structures in a straightforward manner [11]. Let us illustrate these ideas through some examples.

III. Examples:

A.) KdV hierarchy:

Let us consider the Lax operator

$$L = p^2 + u(x), \quad (13)$$

then, it is straightforward to calculate (remember the projection is with respect to the star product)

$$\left(L^{\frac{3}{2}} \right)_+ = p^3 + \frac{3}{2} u \star p - \frac{2\kappa}{2} u^{(1)}, \quad (14)$$

where $u^{(1)} = \frac{\partial u}{\partial x}$.

Therefore, the Moyal-Lax equation

$$\frac{\partial L}{\partial t} = \left\{ L, \left(L^{\frac{3}{2}} \right)_+ \right\}_{\kappa}, \quad (15)$$

gives

$$\frac{\partial u}{\partial t} = - \left(\kappa u^3 + \frac{3}{2} u u^{(1)} \right), \quad (16)$$

which is the KdV equation and the connection between the parameter of non-commutativity and the central charge begins to emerge. However, we will see a more direct relation when we calculate the Hamiltonian structure. The conserved quantities can be determined in a straightforward manner and the first few have the forms

$$\begin{aligned} H_1 &= \text{Tr} L^{\frac{1}{2}} = \int dx \frac{u}{2}, \\ H_2 &= \text{Tr} L^{\frac{3}{2}} = \int dx \frac{u^2}{4}, \\ H_3 &= \text{Tr} L^{\frac{5}{2}} = \int dx (4\kappa u^{(2)}u + u^3). \end{aligned} \quad (17)$$

The commutativity of the flows follows directly from the Moyal-Lax representation [11].

Let us next turn to the question of Hamiltonian structures. First, we define the dual to the Lax operator in (13) by

$$Q = p^{-2} \star q_{-2} + p^{-1} \star q_{-1}, \quad (18)$$

which allows us to define linear functionals as

$$\begin{aligned} F_Q(L) &= \text{Tr} LQ = \int dx uq_{-1}, \\ F_V(L) &= \text{Tr} LV = \int dx uv_{-1}. \end{aligned} \quad (19)$$

then, one can define, in a standard manner, the first two Hamiltonian structures of the system as

$$\begin{aligned} \{F_Q(L), F_V(L)\}_1 &= \text{Tr} L \star \{Q, V\}_\kappa, \\ \{F_Q(L), F_V(L)\}_2 &= \text{Tr} \left((\{L, Q\}_\kappa)_+ \star (L \star V) + (Q \star L)_+ \star \{L, V\}_\kappa \right) \\ &\quad + \frac{1}{2} \int dx \left(\int^x \text{Res}\{Q, L\}_\kappa \right) (\text{Res}\{V, L\}_\kappa). \end{aligned} \quad (20)$$

A direct calculations yields

$$\begin{aligned} \{u(x), u(y)\}_1 &= 2 \frac{\partial}{\partial x} \delta(x - y), \\ \{u(x), u(y)\}_2 &= \left(u(x) \frac{\partial}{\partial x} + \frac{\partial}{\partial x} u(x) + 2\kappa^2 \frac{\partial^3}{\partial x^3} \right) \delta(x - y), \end{aligned} \quad (21)$$

which are indeed the two Hamiltonian structures of KdV. Furthermore, the parameter of non-commutativity, κ , is now seen to be directly related to the central charge of the second Hamiltonian structure, which is known to be the Virasoro algebra.

B.) Two boson hierarchy:

Let us next consider the Lax operator

$$L = p - J_0 + p^{-1} \star J_1. \quad (22)$$

it is easy to calculate (projection with respect to the star product)

$$(L^2)_{\geq 1} = p^2 - 2J_0 \star p, \quad (23)$$

which, then yields from the nonstandard Moyal-Lax equation ²

$$\frac{\partial L}{\partial t} = \left\{ (L^2)_{\geq 1}, L \right\}_{\kappa}, \quad (24)$$

the two boson equations

$$\begin{aligned} \frac{\partial J_0}{\partial t} &= (2J_0 + J_0^2 - 2\kappa J_0')', \\ \frac{\partial J_1}{\partial t} &= (2J_0 J_1 + 2\kappa J_1')', \end{aligned} \quad (25)$$

where the prime denotes derivative with respect to x . Once again there is already a hint of the relation between the parameter κ and the central charge which we will see more explicitly soon. The conserved quantities of the system are defined as

$$H_n = \text{Tr} L^n, \quad (26)$$

and the first few have the explicit forms

$$\begin{aligned} H_1 &= \int dx J_1, \\ H_2 &= -2 \int dx J_0 J_1, \\ H_3 &= 3 \int dx (J_1^2 + J_0^2 J_1 - 2\kappa J_0' J_1). \end{aligned} \quad (27)$$

To study the Hamiltonian structures we define the dual to L as

$$Q = q_0 + q_{-1} \star p^{-1}, \quad (28)$$

so that the linear functionals take the forms

$$F_Q(L) = \text{Tr} LQ = \int dx (q_0 J_1 - q_{-1} J_0). \quad (29)$$

The first two Hamiltonian structures can now be defined in a straightforward manner and have the forms

$$\left\{ F_Q(L), F_V(L) \right\}_1 = \text{Tr} L \star \left\{ Q, V \right\}_{\kappa}, \quad (30)$$

$$\begin{aligned} \left\{ F_Q(L), F_V(L) \right\}_2 &= \text{Tr} \left((\left\{ L, Q \right\}_{\kappa})_+ \star (L \star V) + (Q \star L)_+ \star \left\{ L, V \right\}_{\kappa} \right) - \\ &\int dx \left(\text{Res} \left\{ Q, L \right\}_{\kappa} \text{Res} (L \star V \star p^{-1}) - \text{Res} \left\{ V, L \right\}_{\kappa} \text{Res} (L \star Q \star p^{-1}) \right) \\ &+ \int dx \left(\int^x \text{Res} \left\{ Q, L \right\}_{\kappa} \right) (\text{Res} \left\{ V, L \right\}_{\kappa}). \end{aligned} \quad (31)$$

²Notice that if we take the projection with respect to the usual product (and not the star product), then, the equation becomes inconsistent.

A straightforward calculation yields

$$\begin{aligned} \left(\begin{array}{c} \left\{ J_0, J_0 \right\}_1 \\ \left\{ J_1, J_0 \right\}_1 \end{array} \quad \begin{array}{c} \left\{ J_0, J_1 \right\}_1 \\ \left\{ J_1, J_1 \right\}_1 \end{array} \right) &= - \left(\begin{array}{cc} 0 & \partial \\ \partial & 0 \end{array} \right) \delta(x-y), \\ \left(\begin{array}{c} \left\{ J_0, J_0 \right\}_2 \\ \left\{ J_1, J_0 \right\}_2 \end{array} \quad \begin{array}{c} \left\{ J_0, J_1 \right\}_2 \\ \left\{ J_1, J_1 \right\}_2 \end{array} \right) &= \left(\begin{array}{cc} 2\partial & \partial J_0 - 2\kappa \partial^2 \\ J_0 \partial + 2\kappa \partial^2 & \partial J_1 + J_1 \partial \end{array} \right) \delta(x-y), \end{aligned} \quad (32)$$

which are the usual Hamiltonian structures of the two boson hierarchy. The second Hamiltonian structure, in particular, is the bosonic limit of the $N = 2$ twisted superconformal algebra (one has to redefine the basis to make an exact identification) [14] and the relation between κ and the central charge of the algebra is now explicit.

It is known that the two boson equation reduces to many other integrable models. Without going into details, let us note that if we identify

$$J_0 = -\frac{q'}{q}, \quad J_1 = \hat{q}q, \quad (33)$$

then the Lax operator in (21) can be rewritten as

$$L = q^{-1} \star \tilde{L} \star q, \quad (34)$$

where

$$\tilde{L} = p + q \star p^{-1} \star \hat{q}. \quad (35)$$

In other words, the two Lax operators L and \tilde{L} are related through a gauge transformation. It is now easy to check that the standard Moyal-Lax equation

$$\frac{\partial \tilde{L}}{\partial t} = \left\{ (\tilde{L})_+^2, \tilde{L} \right\}_\kappa, \quad (36)$$

leads to the non-linear Schrödinger equation while, with the identification $\hat{q} = q$, the equation

$$\frac{\partial \tilde{L}}{\partial t} = \left\{ (\tilde{L})_+^3, \tilde{L} \right\}_\kappa, \quad (37)$$

yields the MKdV equation.

IV. Moyal-Lax representation as a Hamiltonian Equation:

The conventional Lax equation (in the standard representation)

$$\frac{\partial L}{\partial t_k} = \left[(L^{\frac{k}{n}})_+, L \right], \quad (38)$$

resembles a Hamiltonian equation with $(L^{\frac{\kappa}{n}})_+$ reminiscent of the Hamiltonian. However, such a relation cannot be further quantified in the language of pseudo-differential operators. In contrast, we will show now that the Moyal-Lax representation has such a natural interpretation.

For concreteness, let us consider an arbitrary flow in the KdV hierarchy described by

$$\frac{\partial L}{\partial t} = \left\{ L, (L^{\frac{2n+1}{2}})_+ \right\}_\kappa. \quad (39)$$

Let us next consider an action of the form

$$S = \int dt \left(p \star \dot{x} - (L^{\frac{2n+1}{2}})_+ \right). \quad (40)$$

The important point to remember, in this, is the fact that $L = L(p, x)$, but does not depend on time explicitly. Thus, we can think of $(L^{\frac{2n+1}{2}})_+$ as the Hamiltonian on the phase space. That this is true follows from the Euler-Lagrange equations of the system, namely

$$\begin{aligned} \dot{x} &= \frac{\partial (L^{\frac{2n+1}{2}})_+}{\partial p} = \left\{ x, (L^{\frac{2n+1}{2}})_+ \right\}_\kappa, \\ \dot{p} &= -\frac{\partial (L^{\frac{2n+1}{2}})_+}{\partial x} = \left\{ p, (L^{\frac{2n+1}{2}})_+ \right\}_\kappa. \end{aligned} \quad (41)$$

These are indeed Hamiltonian equations with Moyal brackets playing the role of Poisson brackets, provided we identify the Hamiltonian of the system with $(L^{\frac{2n+1}{2}})_+$. It also follows now that, since L is a function on this phase space,

$$\frac{\partial L}{\partial t} = \left\{ L, (L^{\frac{2n+1}{2}})_+ \right\}_\kappa. \quad (42)$$

Namely, the Moyal-Lax equation is indeed a Hamiltonian equation with $(L^{\frac{2n+1}{2}})_+$ playing the role of the Hamiltonian. Furthermore, the Moyal-Lax equation, as we have shown, can be derived from an action. Furthermore, although we have shown this for a standard Moyal-Lax representation, it is clear that this derivation will go through for nonstandard representations as well.

V. Hamiltonian Structures for Dispersionless Systems:

The Moyal-Lax representation, of course, has the built in advantage that one can go to the dispersionless limit of an integrable system by simply taking the limit $\kappa \rightarrow 0$. While the Lax representations for various dispersionless integrable models are known [15]-[17], the determination of the Hamiltonian structures (at least the second) from such a Lax function has remained an open question. The Moyal-Lax representation provides a solution to this problem in a natural way. Let us illustrate this with two examples.

First, let us consider the KdV hierarchy, which in the dispersionless limit, goes over to the Riemann hierarchy [15]. With

$$\begin{aligned} L &= p^2 + u, & Q &= p^{-2}q_{-2} + p^{-1}q_{-1}, \\ F_Q(L) &= \text{Tr } LQ = \int dx \, uq_{-1}, \end{aligned} \quad (43)$$

we note that the definition of the first two Hamiltonian structures (19) reduces, in the dispersionless limit, to ($\kappa \rightarrow 0$)

$$\begin{aligned} \{F_Q(L), F_V(L)\}_1 &= \text{Tr } L\{Q, V\}, \\ \{F_Q(L), F_V(L)\}_2 &= \text{Tr } \left((\{L, Q\})_+ LV + (QL)_+ \{L, V\} \right) + \\ &\quad \frac{1}{2} \int dx \left(\int^x \text{Res } \{Q, L\} \right) (\text{Res } \{V, L\}). \end{aligned} \quad (44)$$

A straightforward calculation leads to

$$\begin{aligned} \{u(x), u(y)\}_1 &= 2 \frac{\partial}{\partial x} \delta(x - y), \\ \{u(x), u(y)\}_2 &= (u(x) + u(y)) \frac{\partial}{\partial x} \delta(x - y). \end{aligned} \quad (45)$$

These are indeed the correct Hamiltonian structures of the Riemann equation and while the first structure was already constructed from the Lax function, the construction of the second structure, from the Lax description, was not known so far [15].

As a second example, let us consider polytropic gas [16] with $\gamma = 2$ which can be thought of as the dispersionless limit of the two boson hierarchy. In such a case, the Lax function has the form

$$L = p + u + vp^{-1}. \quad (46)$$

Defining the dual and the linear functional as

$$\begin{aligned} Q &= q_0 + q_{-1}p^{-1}, \\ F_Q(L) &= \text{Tr } LQ = \int dx \, (uq_{-1} + vq_0), \end{aligned} \quad (47)$$

we note that the definition of the first two Hamiltonian structures follows from the dispersionless limit $\kappa \rightarrow 0$ of eqs. (30)-(31) to be

$$\begin{aligned} \{F_Q(L), F_V(L)\}_1 &= \text{Tr } L\{Q, V\}, \\ \{F_Q(L), F_V(L)\}_2 &= \text{Tr } \left((\{L, Q\})_+ LV + (QL)_+ \{L, V\} \right) - \\ &\quad \int dx \left(\text{Res } \{Q, L\} \text{Res}(LVp^{-1}) - \text{Res } \{V, L\} \text{Res}(LQp^{-1}) \right) \\ &\quad + \frac{1}{2} \int dx \left(\int^x \text{Res } \{Q, L\} \right) (\text{Res } \{V, L\}). \end{aligned} \quad (48)$$

A simple calculation yields

$$\begin{aligned} \left(\begin{array}{c} \{u, u\}_1 \\ \{v, u\}_1 \end{array} \quad \begin{array}{c} \{u, v\}_1 \\ \{v, v\}_1 \end{array} \right) &= - \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \delta(x - y), \\ \left(\begin{array}{c} \{u, u\}_2 \\ \{v, u\}_2 \end{array} \quad \begin{array}{c} \{u, v\}_2 \\ \{v, v\}_2 \end{array} \right) &= \begin{pmatrix} 2\partial & \partial u \\ u\partial & \partial v + v\partial \end{pmatrix} \delta(x - y). \end{aligned} \quad (49)$$

These are indeed the correct Hamiltonian structures of this model. We would like to emphasize that it was not known so far how to derive the second Hamiltonian structure from the Lax description of the system [16]. We would also like to note here that the Lax function for the polytropic gas with arbitrary, large γ [16] is highly constrained. Consequently the formulae in (48) need to be modified further (even for the first Hamiltonian structure) and we have not yet analyzed this question.

To summarize, we have studied the properties of Moyal-Lax representation systematically in this letter. In addition to the fact that they allow a smooth passage to the dispersionless models, we have shown that the parameter of non-commutativity is related to the central charge of the second Hamiltonian structure of the system. We have shown that the Moyal-Lax equation can be interpreted as a Hamiltonian equation and can be derived from an action. We have also shown how the Moyal-Lax description leads in a natural manner, in the dispersionless limit, to the Hamiltonian structures of dispersionless integrable models which has been an open question for sometime. In many ways, this alternate description of integrable systems seems more desirable. Properties of the Moyal-Lax representation for supersymmetric integrable systems will be described separately [18].

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